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RAN-2203000205023004**B.Sc. (Sem.-V) Examination November - 2023****Mathematics (Paper - MTH-504)****Real Analysis - II****[Total Marks: 50****सूचना : / Instructions**

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नीचे दशविवेक निशानीवाणी विगतो उत्तरवली पर अवश्य लपववी.
Fill up strictly the details of signs on your answer book

Name of the Examination:

B.Sc. (Sem.-V)

Name of the Subject :

Mathematics (Paper - MTH-504) Real Analysis - II

Subject Code No.: 2203000205023004

Seat No.:

Student's Signature

- (2) All questions are compulsory.
- (3) Follow usual notations.
- (4) Figures to the right indicate marks of the question.
- (5) Total marks 50.

Que: 1 Answer any FIVE as directed.**(10)**

- (1) Let x be a function on \mathbb{R} as $(x) = \begin{cases} x^2 + 2x, & \text{if } x \neq 3 \\ 15, & \text{if } x = 3 \end{cases}$.

Decide the continuity of the function at $x = 3$.

- (2) If $|x - 2| < \delta$, then prove that $\left| \frac{x-2}{x+3} \right| < \frac{\delta}{4}$.
- (3) Define limit of a sequence of points in a metric space.
- (4) Prove that every Cauchy sequence in R_d is convergent.
- (5) Give an illustration of a Cauchy sequence which does not converge in a metric space $\langle M, \rho \rangle$.
- (6) Define equivalence metrics.
- (7) Prove that $B_A[a; r] = A \cap B_M[a; r]$, $A \subset M$ and $a \in M$.
- (8) Let $M = [0, 1]$. Find an open ball about $a = \frac{1}{4}$ with radius = $\frac{1}{2}$.

Que:2 Attempt any TWO. (10)

(1) Define metric space.

Let $M = \mathbb{R}$. Define a function $\rho : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ as $\rho(x, y) = |x - y|$;
 $\forall x, y \in \mathbb{R}$. Prove that $\langle \mathbb{R}, \rho \rangle$ is a metric space.

(2) Let \mathbb{R} be a set of real numbers. Let $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) / x_1, x_2, \dots, x_n \in \mathbb{R}\}$.

Define a function $\rho : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$ as $\rho(x, y) = \left[\sum_{k=1}^n (x_k - y_k)^2 \right]^{\frac{1}{2}}$;

where $x = \langle x_1, x_2, \dots, x_n \rangle$, $y = \langle y_1, y_2, \dots, y_n \rangle$ Prove that $\langle \mathbb{R}^n, \rho \rangle$ is a metric space.

(3) Define $\tau(P, Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ for $P = \langle x_1, y_1 \rangle$ and $Q = \langle x_2, y_2 \rangle$.

Show that τ is a metric for \mathbb{R} .

Que:3 Attempt any TWO. (10)

(1) Let ρ and σ be two metrics for metric space M . If $\exists k > 1$, such that $\frac{1}{k} \sigma(x, y) \leq \rho(x, y) \leq k \sigma(x, y)$; $\forall x, y \in M$, then prove that ρ and σ are equivalent.

(2) Show that a sequence $\{\langle x_n, y_n \rangle\}_{n=1}^{\infty}$ converges to $\langle x, y \rangle$ in \mathbb{R}^2 if and only if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converges in \mathbb{R}^1 to x and y respectively.

(3) Let $\langle M, \rho \rangle$ be a metric space and let a be a point in M . Let f and g be real valued functions whose domains are subsets of M .

If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = N$, then prove that

(a) $\lim_{x \rightarrow a} [f(x) + g(x)] = L + N$ and

(b) $\lim_{x \rightarrow a} [f(x) - g(x)] = L - N$.

Que:4 Attempt any TWO. [10]

(1) Let $\langle M, \rho \rangle$ be a metric space. Let f and g be two real valued functions defined on M . If f and g both are continuous at $a \in M$, then prove that $f - g$. Furthermore, if $g(x) \neq 0$, $x \in M$, then prove that $\frac{f}{g}$ is also continuous at $a \in M$.

- (2) (a) Prove that every function from R_d (into a metric space) is continuous on R_d .
- (b) Let $\langle M, \rho \rangle$ be a metric space. Let $f: M \rightarrow R_d$ be a function. If f is continuous at $a \in M$, then show that $\{a\}$ is an open ball in M .
- (3) Prove that the real valued function f defined on \mathbb{R}^1 and continuous at $a \in \mathbb{R}^1$ if and only if $\lim_{n \rightarrow \infty} x_n = a \Rightarrow \lim_{n \rightarrow \infty} f(x_n) = f(a)$.

Que:5 Attempt any TWO.

[10]

- (1) Define open set in M . Let $M = \mathbb{R}^1$, $a, b \in \mathbb{R}^1$, then prove that (a, b) is open in \mathbb{R}^1 . Also prove that $\{a\}$ is not open in \mathbb{R}^1 .
- (2) Prove that a countable union of countable numbers of open subsets of a metric space M is open.
- (3) If A and B are open subsets of \mathbb{R}^1 , then prove that $A \times B$ is open in \mathbb{R}^2 .
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